Modern Physics Letters A, Vol. 5, No. 30 (1990) 2503–2513


# THE HIDDEN KAC-MOODY SYMMETRY OF THE GEOMETRIC ACTIONS

H. ARATYN*

*Department of Physics, University of Illinois at Chicago, Box 4348, Chicago, IL 60680, USA*

E. NISSIMOV† and S. PACHEVA†

*Department of Physics, Weizmann Institute of Science, Rehovot 76100, Israel*

Received 23 July 1990

A general formalism is proposed to study infinite-dimensional Noether symmetries in arbitrary field theories on group coadjoint orbits as well as in their gauged versions (coset geometric models). The basic tools are generalized group composition laws valid for any geometric action. As a main application, we present a general scheme for constructing the "hidden" Kac-Moody currents.

Models exhibiting infinite-dimensional groups of Noether symmetries in $D$=2 (and higher) space-time dimensions play an increasing role in a variety of areas of modern field theory and statistical mechanics.[1] One of the most fundamental feature of these models is that they can be formulated as dynamical systems on coadjoint orbits[2] of infinite-dimensional Lie groups (for a formulation appropriate for path-integral quantization, see Refs. 3 – 5) or as gauged versions of the latter.[5-7]

We start with a brief recapitulation of the basics of the group coadjoint orbit method. In Sec. 2 we present the general group composition law for any geometric action constructed by the co-orbit method on infinite-dimensional Lie groups. The composition laws are then applied to describe the associated "hidden" Kac-Moody symmetries, as well as to explicitly construct all conserved currents. Finally, in the last section the method is extended to geometric actions on coset spaces and models obtained by gauging the underlying co-orbit model. The latter is equivalent to the Hamiltonian reduction procedure (see e.g. Ref. 8).

## 1. Geometric Actions on Group Coadjoint Orbits

The elements of an infinite-dimensional Lie algebra with central extension $\tilde{\mathcal{G}} = \mathcal{G} + \mathbb{R}$ are represented as pairs ($\xi \in \mathcal{G}, n \in \mathbb{R}$):

$$(\xi, n) \equiv \int dx\; \xi(x)\mathcal{T}(x) + n\hat{I}\,, \tag{1}$$

* Work supported in part by U. S. Department of Energy, contract No. DE-FG02-84ER40173.

† On leave from: Institute of Nuclear Research and Nuclear Energy, Boul. Lenin 72, 1784 Sofia, Bulgaria.

$$[\mathcal{T}(x), \mathcal{T}(x')] = \int dx'' f(x,x';x'')\,\mathcal{T}(x'') + \omega(x,x')\,\hat{I}\,, \tag{2}$$

$$[(\xi_1, n_1), (\xi_2, n_2)] = ([\xi_1, \xi_2], \omega(\xi_1, \xi_2))\,, \tag{3}$$

where $(\mathcal{T}(x), \hat{I})$ denote the generators of $\tilde{\mathcal{G}}$, $x$ is a continuous label including possibly discrete indices (as in the case of Kac-Moody algebras) and $\omega(x, x')$ denotes a non-trivial $\mathcal{G}$-two-cocycle ("anomaly" in physicist's language). Let $\tilde{\mathcal{G}}^*$ with elements $(B, c) \equiv \int dx\, B(x)\, \mathcal{T}^*(x) + c\, \hat{I}^*$ be the dual space of $\tilde{\mathcal{G}}$ with respect to the extension of the natural bilinear form $\langle \cdot \mid \cdot \rangle$ on $\mathcal{G} \times \mathcal{G}^*$:

$$\langle (B,c) \mid (\xi, n) \rangle = \langle B \mid \xi \rangle + cn \equiv \int dx\, B(x)\, \xi(x) + cn \tag{4}$$

$\mathcal{T}^*(x)$, $\hat{I}^*$) ( describes a basis in the vector space $\tilde{\mathcal{G}}^*$orthonormal to the basis $(\mathcal{T}(x), \hat{I})$ in $\tilde{\mathcal{G}}$ with respect to the canonical bilinear form (4).

Let us denote by $G$ the infinite-dimensional Lie group associated with $\mathcal{G}$. The adjoint actions of $G$ and $\mathcal{G}$ on $\mathcal{G}$, given by $\mathrm{Ad}(g)\xi = g\xi g^{-1}$ and $\mathrm{ad}(\xi)\eta = [\xi, \eta]$, and the corresponding coadjoint actions on $\mathcal{G}^*$ ($\langle \mathrm{Ad}^*(g)B \mid \xi \rangle = \langle B \mid \mathrm{Ad}(g^{-1})\xi \rangle$ and $\langle \mathrm{ad}^*(\xi)\, B \mid \eta \rangle = -\langle B \mid [\xi, \eta] \rangle$ induced by (4)) can be extended to $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}^*$, respectively:

$$\tilde{\mathrm{A}}\mathrm{d}(g)(\xi, n) = (\mathrm{Ad}(g)\,\xi, n + \lambda\, \langle S(g^{-1}) \mid \xi \rangle)\,, \tag{5}$$

$$\tilde{\mathrm{A}}\mathrm{d}^*(g)(B,c) = (\mathrm{Ad}^*(g)\,B + c\lambda S(g), c) \equiv (B + \Sigma_{(B,c)}(g), c)\,, \tag{6}$$

$$\tilde{\mathrm{a}}\mathrm{d}^*(\xi, n)(B,c) = (\mathrm{ad}^*(\xi)\,B + c\lambda s(\xi), 0) \equiv (\sigma_{(B,c)}(\xi), 0)\,, \tag{7}$$

where we introduced for future convenience $\Sigma_{(B,c)}(g), \equiv c\lambda S(g) + \mathrm{Ad}^*(g)\,B - B$.

In (5) – (7) $\lambda$ denotes a normalization constant specific for the model. $S(g)$ in Eqs. (5) and (6) is a $\mathcal{G}^*$-valued one-cocycle on $G$:

$$(\delta S)\,(g_1, g_2) \equiv \mathrm{Ad}^*(g_1)\, S(g_2) - S(g_1 g_2) + S(g_1) = 0\,. \tag{8}$$

Similarly, $\Sigma_{(B,c)}(g)$ is a $\mathcal{G}^*$-valued $G$-cocycle as well:

$$\Sigma_{(B,c)}(g_1 g_2) = \Sigma_{(B,c)}(g_1) + \mathrm{Ad}^*(g_1)\, \Sigma_{(B,c)}(g_2)\,, \tag{9}$$

$S(g)$ can be obtained directly from the Lie-algebra $\mathcal{G}$- cocycle

$$\omega(\xi, \eta) = \int dx dx'\, \xi(x)\, \omega(x,x')\, \eta(x') = -\lambda \langle s(\xi) \mid \eta \rangle \tag{10}$$

(cf. (2)) through the basic Kirillov's formula[9]:

$$\omega(\mathrm{Ad}(g^{-1})\,\xi, \mathrm{Ad}(g^{-1})\,\eta) - \omega(\xi, \eta) = \lambda\, \langle S(g) \mid [\xi, \eta] \rangle\,. \tag{11}$$

According to Eq. (5), $S(g)$ is the "integrated anomaly", i.e., the "anomaly" for finite group transformations $g \in G$. $s(\cdot)$ and $\sigma_{(B,c)}(\cdot)$ in Eq. (7) are the infinitesimal forms of $S(g)$ and $\Sigma_{(B,c)}(g)$, given respectively by

$$S(I + \varepsilon) = s(\varepsilon) + O(\varepsilon^2), \varepsilon \in \mathcal{G}\,; \quad \Sigma_{(B,c)}(I + \varepsilon) = \sigma_{(B,c)}(\varepsilon) + O(\varepsilon^2)\,. \tag{12}$$

After constructing $S(g)$, one introduces the fundamental $\mathcal{G}$-valued one-form $y(g)$ on $\mathcal{G}^*$ as functional of $S(g)$ through the following equation,

$$dS(g) = \mathrm{ad}^*(y(g))\, S(g) + s(y(g)). \tag{13}$$

The integrability condition for (13) implies that $y(g)$ is a Maurer-Cartan one-form and, correspondingly, the cocycle property (8) implies a similar cocycle property for $y(g)$:

$$dy(g) = \frac{1}{2}[y(g), y(g)], \tag{14}$$

$$y(g_1 g_2) = y(g_1) + \mathrm{Ad}(g_1)\, y(g_2). \tag{15}$$

It was shown in Ref. 10 that, on each $G$-coadjoint orbit

$$\mathcal{O}_{(B_0,c)} \equiv \{\tilde{\mathrm{A}}\mathrm{d}^*(g)(B_0, c); \forall g \in G\} \tag{16}$$

endowed with the natural Kirillov-Kostant-Souriau (KKS) symplectic structure,[2] one can write the corresponding geometric action as follows,

$$\begin{aligned} W_{(B_0,c)}[g] &= -\lambda c \int \left[ \langle S(g) \,|\, y(g) \rangle - \frac{1}{2} d^{-1}(\langle s(y(g)) \,|\, y(g) \rangle) \right] \\ &\quad - \int \langle \mathrm{Ad}^*(g) B_0 \,|\, y(g) \rangle = -\int \Bigg[ \langle \Sigma_{(B_0,c)}(g) \,|\, y(g) \rangle \\ &\quad - \frac{1}{2} d^{-1}\left(\langle \sigma_{(B_0,c)}(y(g)) \,|, y(g) \rangle\right) \Bigg]. \end{aligned} \tag{17}$$

Equivalently, using (8) and (15), Eq. (17) can be rewritten as[a]

$$\begin{aligned} W_{(B_0,c)}[g] &= -\frac{1}{2} c\lambda \int d^{-1}(\langle s(y(g^{-1})) \,|\, y(g^{-1}) \rangle) + \int \langle B_0, y(g^{-1}) \rangle \\ &= -\frac{1}{2} \int d^{-1}(\langle \sigma_{(B_0,c)}(y(g^{-1})) \,|\, y(g^{-1}) \rangle). \end{aligned} \tag{18}$$

In (17) and (18), the integral is over an one-dimensional curve on the phase space $\mathcal{O}_{(B_0,c)}$ with parameter $t$. Accordingly, the exterior derivative along the curve becomes $d = dt\partial_t$. In what follows we shall use the notations $t \equiv x^+$, $x \equiv x^-$ to agree with the usual notation in $D = 2$ conformal field theory.

## 2. Group Composition Laws and Noether Symmetries

Applying the group-cocycle relations (8) and (15) to the specific form of the geometric action (17), we obtain the following general form of the Polyakov-Wiegmann composition laws[11] for arbitrary group co-orbit models,

$$W_{(B_0,c)}[g_1 g_2] = W_{(B_0,c)}[g_1] + W_{(B_0,c)}[g_2] + \int \langle \Sigma_{(B_0,c)}(g_2) \,|\, y(g_1^{-1}) \rangle. \tag{19}$$

[a] This is the form of the geometric action proposed in Ref. 4.

For the particular class of co-orbits $\mathcal{O}_{(B_0,c)}$ with $B_0 = 0$, Eq. (19) was derived in Ref. 12.

As non-trivial examples of the power of the general composition laws (19), let us consider the case of $(N, 0)$ super-Virasoro group with elements being $N$-extended superconformal diffeomorphisms (see, e.g., Ref. 13):

$$g \leftrightarrow (\tilde{X} = \tilde{X}(x^-, \theta^j), \tilde{\Theta}^i = \tilde{\Theta}^i(x^-, \theta^j))$$

$$D^{(i}\tilde{\Theta}_k D^{j)}\tilde{\Theta}^k - \delta^{ij}\frac{1}{N}(D^l\tilde{\Theta}^k D_l\tilde{\Theta}_k) = 0\,, \quad D^i\tilde{X} - i\tilde{\Theta}^k D^i\tilde{\Theta}_k = 0\,,$$

$$D^i \equiv \frac{\partial}{\partial\theta_i} - i\theta^i\partial_-\,, \quad \{D^i, D^j\} = 2i\delta^{ij}\partial_-\,, \quad i, j = 1, \ldots, N\,. \tag{20}$$

The elements $\xi$ of the Lie algebra of infinitesimal superconformal diffeomorphisms are $(N, 0)$ superfields $\xi = \xi(x^-, \theta^i)$ with a commutator (3) $[\xi_1, \xi_2] = \xi_1\partial_-\xi_2 - (\partial_-\xi_1)\xi_2 - i/2\,(D^j\xi_1)(D_j\xi_2)$.

Both the Maurer-Cartan form[14,15]

$$y(g) \leftrightarrow y(\tilde{X}, \tilde{\Theta}^i) = N\frac{d\tilde{X} + i\tilde{\Theta}^k d\tilde{\Theta}_k}{(D^i\tilde{\Theta}^j)(D_i\tilde{\Theta}_j)} \tag{21}$$

and the "integrated anomaly" $S(g) \leftrightarrow S(\tilde{X}, \tilde{\Theta}^i)$, the super-Schwarzian,[13] are known. In particular, the infinitesimal form of $S(\tilde{X}, \tilde{\Theta}^i)$ reads

$$s_{N=1}(\xi) = \partial_-^2 D\xi\,, \quad s_{N=2}(\xi) = \frac{1}{2}\varepsilon_{ij}D^iD^j\partial_-\xi\,, \quad s_{N=3}(\xi) = \frac{1}{3}\varepsilon_{ijk}D^iD^jD^k\xi\,. \tag{22}$$

Plugging (21) and $S(\tilde{X}, \tilde{\Theta}^i)$ into (19) we immediately obtain the $(N, 0)$ supergravity composition laws. Composition laws have been previously derived for the usual $D = 2$ gravity[6,7] and for simple $N = 1$ supergravity.[16]

Equation (19) is a succinct expression of all pertinent Noether symmetries[12,17] and is a crucial ingredient for finding the exact solution of the quantum Ward identities.

The infinitesimal form of (19) for $g_1 = I + \varepsilon$, $\varepsilon \in \mathcal{G}$, $g_2 = g$,

$$\delta^l_\varepsilon W_{(B_0,c)}[g] \equiv W_{(B_0,c)}[(I+\varepsilon)g] - W_{(B_0,c)}[g] = -\int \langle \Sigma_{(B_0,c)}(g) \,|\, d\varepsilon \rangle \tag{23}$$

yields the conserved current $\Sigma_{(B_0,c)}(g)$ whose Poisson bracket algebra exactly coincides (up to a sign) with the original algebra[17,12] $\tilde{\mathcal{G}}$:

$$d\Sigma_{(B_0,c)}(g)\big|_{\text{on-shell}} = 0\,, \tag{24}$$

$$\left\{\langle \Sigma_{(B_0,c)}(g) | \xi \rangle\,, \langle \Sigma_{(B_0,c)}(g) | \eta \rangle\right\}_{\text{PB}}^{x^+ = x^{+\prime}} = -\langle \Sigma_{(B_0,c)}(g) | [\xi, \eta] \rangle + \langle \sigma_{(B_0,c)}(\xi) | \eta \rangle\,. \tag{25}$$

Equation (25) is a consequence of the infinitesimal form of the cocycle property (9) of $\Sigma_{(B_0,c)}(g)$:

$$\delta^l_\varepsilon \Sigma_{(B_0,c)}(g) = \sigma_{(B_0,c)}(\varepsilon) + \mathrm{ad}^*(\varepsilon)\Sigma_{(B_0,c)}(g). \tag{26}$$

Let us point out that Eq. (25) is the form of the fundamental KKS Poisson struture[2] on the co-orbit $\mathcal{O}_{(B_0,c)}$.

Using (11) one easily obtains the relation

$$d\Sigma_{(B_0,c)}(g) = -\,\mathrm{Ad}^*(g)\,\sigma_{(B_0,c)}(y(g^{-1})), \tag{27}$$

showing that on-shell

$$\sigma_{(B_0,c)}(y(g^{-1})) \equiv c\lambda s(y(g^{-1})) + \mathrm{ad}^*(y(g^{-1}))\,B_0 = 0 \tag{28}$$

due to (24). This group-covariant form of the equations of motion can be equivalently derived by considering the infinitesimal form of (19) for the right group translation, i.e., $g_1 \equiv g,\ g_2 = I + \eta,\ \eta \in \mathcal{G}$:

$$\delta^r_\eta W_{(B_0,c)}[g] = \int \langle \sigma_{(B_0,c)}(\eta) \,|\, y(g^{-1})\rangle. \tag{29}$$

As already discussed in Ref. 12, Eq. (29) yields the Noether theorem for the symmetry of the action (17) under the right group translations $g \to g(I+\eta_0)$ where $\eta_0 \in \mathcal{G}_{\mathrm{stat}}$, the Lie algebra of the stationary subgroup $G_{\mathrm{stat}}$ of the orbit $\mathcal{O}_{(B_0,c)}$. Let us recall that according to Eqs. (6) and (7):

$$G_{\mathrm{stat}} = \{g \in G;\ \Sigma_{(B,c)}(g) = 0\}, \tag{30}$$

$$\mathcal{G}_{\mathrm{stat}} = \{\eta_0 \in \mathcal{G};\ \sigma_{(B,c)}(\eta_0) = 0\}. \tag{31}$$

For simplicity, let us concentrate on the class of orbits $\mathcal{O}_{(B_0,c)}$ with $B_0 = 0$ which underlie most physically interesting models. The generic form of $\eta_0 \in \mathcal{G}_{\mathrm{stat}}$ (Eq. (31) for $B_0 = 0$) reads

$$\eta_0 = \mathcal{L}^A(x^-)\,\eta_A(x^+), \quad s(\mathcal{L}^A(x^-)) = 0, \tag{32}$$

$$[\mathcal{L}^A(\cdot), \mathcal{L}^B(\cdot)]_{\mathcal{G}} \equiv \int dx'\,dx''\,f(x',x'';x^-)\,\mathcal{L}^A(x')\,\mathcal{L}^B(x'') = f^{AB}_C\,\mathcal{L}^C(x^-). \tag{33}$$

Here $\eta_A$ corresponds to the set of group parameters of the finite-dimensional $G_{\mathrm{stat}}$, $f^{AB}_C$ denote its structure constants and the subscript "$\mathcal{G}$" in (33) stresses that the commutator is in the original infinite-dimensional algebra $\mathcal{G}$, Eq. (2). Therefore, the Noether theorem together with (29):

$$\delta^r_{\eta_0} W[g] = c\lambda \int \langle s(\mathcal{L}^A(x^-)\,\eta_A(x^+,x^-) \,|\, y(g^{-1})\rangle = \int d^2x\,\partial_-\eta_A K^A(g) \tag{34}$$

yields the *off-shell* equation determining the conserved Noether current $K^A$ as a linear function of $y_+(g^{-1})$ and its derivative with respect to $x^-$ (recall $y(g^{-1}) \equiv dx^+ y_+(g^{-1})$):

$$c\lambda\,\mathcal{L}^A(x^-)\,s(y_+(g^{-1})) - \partial_- K^A(g) = 0. \tag{35}$$

Comparing (35) with (28) (for $B_0 = 0$) one concludes that on-shell

$$s(y_+(g^{-1})) = 0 \leftrightarrow \partial_- K^A(g) = 0. \tag{36}$$

Using the identity

$$\langle s(\mathcal{L}^A \eta_A) | \mathcal{L}^B \xi_B \rangle = \int dx^- \partial_- \eta_A \gamma^{AB} \xi_B \,, \tag{37}$$

where $\gamma^{AB}$ denotes the Killing metric of $\mathcal{G}_{\rm stat}$, we can invert Eq. (35):

$$c\lambda y_+(g^{-1}) = \mathcal{L}^A(x^-) K_A(g) \,. \tag{38}$$

Now, one can easily show that the infinitesimal form of the cocycle condition (15) ($g_1 = I - \eta$, $g_2 = g^{-1}$)

$$\delta^r_\eta y_+(g^{-1}) = -\partial_+ \eta + [y_+(g^{-1}), \eta] \tag{39}$$

together with (38) imply the existence of a classical $G_{\rm stat}$ Kac-Moody current algebra for any group $G$ co-orbit model. This can be seen as follows. Consider (39) for $\eta = \eta_0 \in \mathcal{G}_{\rm stat}$ and recall that $\int dx^{+\prime} \eta_A(x^{+\prime}) \gamma^{AB} K_B(g; x^{+\prime}, x^-)$ is the generator of the corresponding symmetry $g \to g(I + \eta)$:

$$\{c\lambda y_+(g^{-1}), \int dx^{+\prime} \eta_A(x^{+\prime}) \gamma^{AB} K_B(g; x^{+\prime}, x^-)\}_{\rm PB}^{(x^- = x^{-\prime})}$$
$$\equiv \delta_{\eta_0} c\lambda y_+(g^{-1}) = -c\lambda \partial_+ \eta_0 + [c\lambda y_+(g^{-1}), \eta_0]_{\mathcal{G}} \,. \tag{40}$$

Inserting (32) and (38) into (40) we get

$$\{K_A(g), K_B(g)\}_{\rm PB}^{(x^- = x^{-\prime})} = -f^C_{AB} K_C(g)\, \delta(x^+ - x^{+\prime}) - c\lambda \gamma_{AB} \partial_+ \delta(x^+ - x^{+\prime}) \,, \tag{41}$$

i.e., the classical "hidden" $G_{\rm stat}$ Kac-Moody current algebra.

As a non-trivial example, let us consider again the $(N, 0)$ super-Virasoro case (Eqs. (20) and (21)), whose stationary subgroup is[13] $G_{\rm stat} = {\rm OSp}(N; 2)$, and take for definiteness $N = 2$.

The functions $\mathcal{L}^A(\cdot)$ spanning OSp(2;2) according to (33) are[b]: 1, $x^-$, $x^{-2}$, $\theta^\pm, \theta^\pm x^-, \theta^- \theta^+$. The relevant OSp(2;2) "hidden" currents $K_A \equiv \{L_\pm, L_0, T_0; G^+_{\pm(1/2)}, G^-_{\pm(1/2)}\}$ are given explicitly in terms of $y_+(\tilde{X}, \tilde{\Theta}, \tilde{\tilde{\Theta}})$ from (21) in the form

$$L_+ = \partial^2_- y_+(\tilde{X}, \tilde{\Theta}, \tilde{\tilde{\Theta}})|_{\theta^\pm = 0} \,,$$
$$L_0 = (x^- \partial^2_- - \partial_-)\, y_+(\tilde{X}, \tilde{\Theta}, \tilde{\tilde{\Theta}})|_{\theta^\pm = 0} \,,$$
$$L_- = ((x^-)^2\, \partial^2_- - 2x^- \partial_- + 2)\, y_+(\tilde{X}, \tilde{\Theta}, \tilde{\tilde{\Theta}})|_{\theta^\pm = 0} \,,$$
$$T_0 = \frac{1}{2}[D^-, D^+]\, y_+(\tilde{X}, \tilde{\Theta}, \tilde{\tilde{\Theta}})|_{\theta^\pm = 0} \,,$$
$$G^\pm_{-(1/2)} = i\partial_- D^\pm y_+(\tilde{X}, \tilde{\Theta}, \tilde{\tilde{\Theta}})|_{\theta^\pm = 0} \,,$$
$$G^\pm_{+(1/2)} = -i(1 + x^- \partial_-)\, D^\pm y_+(\tilde{X}, \tilde{\Theta}, \tilde{\tilde{\Theta}})|_{\theta^\pm = 0} \,. \tag{42}$$

Let us also show that the canonical Noether energy-momentum tensor $T_{++}$ automatically has a Sugawara form in terms of the "hidden" conserved currents

[b] For (2,0) superspace it is convenient to use complex U(1) instead of real SO(2), notations: $\theta^\pm = 1/\sqrt{2}\,(\theta^1 \pm i\theta^2)$, $D^\pm = 1/\sqrt{2}\,(D^1 \pm iD^2)$.

$K_A(g)$, Eqs. (38) and (41). Indeed, under infinitesimal $x^+$-reparametrization $x^+ \to x^+ + \rho^+(x^+)$, $y_+(g^{-1})$ transforms as

$$\delta_\rho y_+(g^{-1}) = \partial_+ (\rho^+ y_+(g^{-1})). \tag{43}$$

Equation (43) is easily seen to be equivalent to an infinitesimal *field-dependent* right group translation

$$\delta_\rho y_+(g^{-1}) = \delta^r_{\eta(\rho)} y_+(g^{-1}) = -\partial_+ \eta(\rho) + [y_+(g^{-1}), \eta(\rho)] \tag{44}$$

with $\eta(\rho) = \rho^+(x^+)\, y_+(g^{-1})$.

Then, the Noether theorem together with (29):

$$\delta_{\eta(\rho)} W[g] = c\lambda \int \langle s(\rho^+(x^+, x^-)\, y_+(g^{-1})) \,|\, y_+(g^{-1}) \rangle = \int dx^2 (\partial_- \rho^+)\, T_{++}(g) \tag{45}$$

yields the *off-shell* relation determining $T_{++}(g)$ as a bilinear function of $y_+(g^{-1})$

$$c\lambda y_+(g^{-1})\, s(y_+(g^{-1})) - \partial_- T_{++}(g) = 0 \tag{46}$$

(with the on-shell conservation of $T_{++}$ being equivalent to the equations of motion $s(y_+(g^{-1})) = 0$, (28), for $B_0 = 0$). Using the identity (37) and relation (38) we obtain the general Sugawara form:

$$T_{++}(g) = \frac{1}{2c\lambda} \gamma^{AB} K_A(g)\, K_B(g). \tag{47}$$

## 3. Coset and Gauged Co-Orbit Models

The previous formalism can be easily generalized to geometric actions on coset spaces and for gauging of group co-orbit models.

Let $H$ be an (infinite-dimensional) subgroup of $G$ and consider the Mackey decomposition (see, e.g., Ref. 18):

$$g = h(g)\, l(g)\,, \qquad h(g) \in H\,. \tag{48}$$

(Formally $g = \exp \int dx\, \alpha(x)\, \mathcal{T}_G(x)$, $h(g) = \exp \int dx\, \alpha(x)\, \mathcal{T}_H(x)$, where $\mathcal{T}_H(x)$ denote the generators of $\mathcal{H}$– the Lie algebra of $H$, i.e., $\mathcal{G} = \mathcal{H} + \mathcal{P}$, $\mathcal{T}_G(x) = \mathcal{T}_H(x) + \mathcal{T}_P(x)$.) Using the generalized composition laws (19) we obtain the following coset action in terms of the action (17) on the group co-orbit $\mathcal{O}_{(B_0,c)}$ of $G$:

$$\begin{aligned} W_{G/H} \equiv W_{(B_0,c)}[h(g)^{-1} g] &= W_{(B_0,c)}[g] - W_{(B_0,c)}[h(g)] \\ &+ \int \langle \Sigma_{(B_0,c)}(g) - \Sigma_{(B_0,c)}(h(g)) \,|\, y(h(g)) \rangle. \end{aligned} \tag{49}$$

From its definition one easily verifies that the coset action (49) is manifestly invariant under the left group translation $g \to h_0 g$ for any $h_0 \in H$. In the particular case of "anomaly-free" subgroup $H$, i.e.,

$$\langle \sigma_{(B_0,c)}(\xi_H) \,|\, \eta_H \rangle = 0 \text{ for any } \xi_H, \eta_H \in \mathcal{H}\,, \tag{50}$$

the coset action (49) simplifies to

$$W_{G/H} = W_{(B_0, c)}[g] + \int \langle \Sigma_{(B_0, c)}(g) \mid y(h(g)) \rangle . \tag{51}$$

Another important generalization of the class of geometric actions on $G$ co-orbits (17) (closely related to the coset model construction above) is gauging of a subgroup $H \subset G$. This is achieved by introducing, into the $G$ co-orbit action, a set of first-class constraints:

$$W^{\text{gauged}}_{(B_0, c)}[g; \Lambda_H] = W_{(B_0, c)}[g] + \int \langle \Sigma_{(B_0, c)}(g) - \Sigma^{(0)} \mid \Lambda_H \rangle . \tag{52}$$

In (52) $\Lambda_H$ is an element of the subalgebra $\mathcal{H}$ and serves as a Lagrange multiplier for the first-class constraints:

$$(\Sigma_{(B_0, c)}(g) - \Sigma^{(0)})|_{\mathcal{H}^*} = 0 \leftrightarrow \langle \Sigma_{(B_0, c)}(g) - \Sigma^{(0)} \mid \xi_H \rangle \equiv \Phi(g; \xi_H) = 0 \tag{53}$$

for arbitrary fixed $\xi_H \in \mathcal{H}$, and $\Sigma^{(0)}$ is a constant element of $\mathcal{G}^*$. Recalling the Poisson bracket algebra of $\Sigma_{(B_0, c)}(g)$, (25), we deduce that in order for $\Phi(g; \xi_H)$, (53), to be first-class, we must have for any $\xi_H$, $\eta_H \in \mathcal{H}$

$$\langle \Sigma^{(0)} \mid [\xi_H, \eta_H] \rangle = 0 , \tag{54}$$

and $\mathcal{H}$ must be "anomaly free" subgroup (cf. Eq. (50)). The first-class Poisson bracket algebra then exactly coincides (up to a sign) with the subalgebra $\mathcal{H}$ (cf. (25)):

$$\{\Phi(g; \xi_H), \Phi(g; \eta_H)\}_{\text{PB}} = -\Phi(g; [\xi_H, \eta_H]) . \tag{55}$$

It is straightforward to show (using (9), (15), (50) and (54)) that the gauged action (52) is invariant under the gauge transformations:

$$g \to hg , \qquad \Lambda_H \to \text{Ad}(h)\, \Lambda_H + y_H(h) \tag{56}$$

for arbitrary $h \in H$ ($y_H(h)$ being the Maurer-Cartan form for the subgroup $H$). Let us also point out that the second term in (52) is a Noether coupling of the gauge field $\Lambda_H$ to the conserved current $\Sigma_{(B_0, c)}(g)$, (24).

Several covariant gauge-fixings are possible. In particular, we may choose

$$\chi(g; \zeta_K) \equiv \langle \Sigma_{(B_0, c)}(g) - \Sigma^{(0)} \mid \zeta_K \rangle = 0 , \tag{57}$$

where $\zeta_K$ is an arbitrary element of $\mathcal{K}$ – another subspace (as a vector space) of $\mathcal{G}$ such that dim $\mathcal{K}$ = dim $\mathcal{H}$. The Faddeev-Popov ghost operator is given by (cf. Eq. (25))

$$\{\Phi(g; \xi_H), \chi(g; \zeta_K)\}_{\text{PB}} = \langle \sigma_{(B_0, c)}(\xi_H) + \text{ad}^*(\xi_H)\, \Sigma_{(B_0, c)}(g) \mid \zeta_K \rangle \equiv \int \xi_H \Delta_{\Phi\Pi}(g)\, \zeta_K . \tag{58}$$

The non-degeneracy of the ghost operator $\Delta_{\Phi\Pi}$, (58), follows from the non-degeneracy (invertibility) of the original KKS Poisson structure (see Eq. (25)).

Introducing a pair of ghosts ($c_H \in \mathcal{H}$, $b_K \in \mathcal{K}$), the BRST gauge-fixed version of (52) reads

$$W^{\mathrm{BRST}}_{(B_0,\,c)}[g;\Lambda_H,\mathcal{M}_K,c_H,b_K]\equiv W_{(B_0,\,c)}[g]$$
$$+\int\langle\Sigma_{(B_0,\,c)}(g)-\Sigma^{(0)}\,|\,\Lambda_H+\mathcal{M}_K\rangle-\int c_H\Delta_{\Phi\Pi}b_K\,. \tag{59}$$

In (59) $\mathcal{M}_K\in\mathcal{K}$ denotes the Lagrange multiplier for the gauge-fixing conditions (57). The BRST transformations leaving (59) invariant read

$$\delta_{\mathrm{BRST}}\,g=(I+\alpha c_H)\,g\,,\ \ \delta_{\mathrm{BRST}}\Lambda_H=\alpha(dc_H-[\Lambda_H,c_H])\,, \tag{60}$$

$$\delta_{\mathrm{BRST}}\,c_H=\frac{1}{2}\,\alpha[c_H,c_H]\,,\ \ \delta_{\mathrm{BRST}}\,b_K=-\,\alpha\mathcal{M}_K\,,\ \ \delta_{\mathrm{BRST}}\mathcal{M}_K=0\,, \tag{61}$$

where $\alpha$ is a constant fermionic parameter.

Now we can easily generalize the fundamental composition law for the $G$ co-orbit action (19) to the case of the BRST gauge-fixed action (59):

$$W^{\mathrm{BRST}}_{(B_0,\,c)}[g_1g_2;\Lambda_H,\mathcal{M}_K,c_H,b_K]=W^{\mathrm{BRST}}_{(B_0,\,c)}[g_1;\Lambda_H,\mathcal{M}_K,c_H,b_K]$$
$$+\,W_{(B_0,\,c)}[g_2]+\int\langle\Sigma_{(B_0,\,c)}(g_2)\,|\,y(g_1^{-1};\Lambda_H,\mathcal{M}_K,c_H,b_K)\rangle\,, \tag{62}$$

where

$$y_{\mathrm{BRST}}(g^{-1};\Lambda_H,\mathcal{M}_K,c_H,b_K)\equiv y(g^{-1})+\mathrm{Ad}(g^{-1})(\Lambda_H+\mathcal{M}_K+[c_H,b_K])\,. \tag{63}$$

The crucial property of the composite field (63) is that it is invariant under BRST transformations (61). Clearly $y_{\mathrm{BRST}}\,(g^{-1};\,\Lambda_H,\,\mathcal{M}_K,\,c_H,\,b_K)$ is the BRST-invariant extension of $y(g^{-1})+\mathrm{Ad}(g^{-1})\,\Lambda_H$, which is gauge invariant object under transformation (56).

As in the non-gauged case, the BRST composition law (62) plays an important role since it contains the whole information about all relevant Noether symmetries. Consider, for instance, infinitesimal right group translations $g\to g(I+\eta)$ in (62) (we consider for simplicity the case of co-orbits $\mathcal{O}_{(B_0,\,c)}$ with $B_0=0$):

$$\delta^r_\eta W^{\mathrm{BRST}}_{(0,\,c)}[g;\Lambda_H,\mathcal{M}_K,c_H,b_K]=c\lambda\int\langle s(\eta)\,|\,y_{\mathrm{BRST}}(g^{-1};\Lambda_H,\mathcal{M}_K,c_H,b_K)\rangle\,. \tag{64}$$

Comparing (64) with (29) and repeating the same arguments following the latter equation, we find that the BRST action (59) is invariant under arbitrary right translations $g\to g(I+\eta_0)$ with $\eta_0\in\mathcal{G}_{\mathrm{stat}}$ (31) – the stationary subalgebra of the $G$ co-orbit $\mathcal{O}_{(B_0=0,\,c)}$, exactly as in the non-gauged case. Hereby, all Eqs. (32) – (40) remain valid with the substitution of $y_{\mathrm{BRST}}\,(g^{-1};\,\Lambda_H,\,\mathcal{M}_K,\,c_H,\,b_K)$ instead of $y(g^{-1})$. Therefore, the action $W^{\mathrm{BRST}}_{(0,\,c)}$ possesses the same "hidden" $G_{\mathrm{stat}}$ Kac-Moody current algebra (41) as the action $W_{(0,\,c)}$ in the non-gauged case. The corresponding currents are given by (cf. Eqs. (38), (36), and (63))

$$y_{\mathrm{BRST}}(g^{-1};\Lambda_H,\mathcal{M}_K,c_H,b_K)=\mathcal{L}^A(x^-)\,K^{\mathrm{BRST}}_A(g;\Lambda_H,\mathcal{M}_K,c_H,b_K)\,, \tag{65}$$

$$s(y_{\mathrm{BRST}}(g^{-1};\Lambda_H,\mathcal{M}_K,c_H,b_K))\,|_{\text{on-shell}}=0\to\partial_-K^{\mathrm{BRST}}_A(g;\Lambda_H,\mathcal{M}_K,c_H,b_K)=0\,. \tag{66}$$

The first equation of (66) is the BRST-invariant equation of motion. The reduced phase space $\mathcal{O}^*$ is defined by strongly imposing on the initial "large" phase space $\mathcal{O}_{(0,c)}$ both the first-class constraints (53) as well as their associated gauge-fixing conditions (57). Let us denote by $\phi(x^-)$ the coordinates on $\mathcal{O}^*$, i.e., the physical degrees of freedom. They enter in $g = g(\phi)$ – the solution of the constraint equations:

$$\langle S(g(\phi)) - S^{(0)} \mid \xi_H \rangle = 0\,, \quad \langle S(g(\phi)) - S^{(0)} \mid \zeta_K \rangle = 0\,. \tag{67}$$

After removing the non-physical degrees of freedom from the gauged co-orbit model (52), we find the following set of physical conserved currents (besides the usual energy-momentum tensor),

$$K_A(\phi) = K_A^{\mathrm{BRST}}(g(\phi); 0, 0, 0, 0)\,, \tag{68}$$

$$\mathcal{J}(\phi) = c\lambda(S(g(\phi)) - S^{(0)})\,|_{\mathcal{P}}\; (\mathcal{G} = \mathcal{H} + \mathcal{K} + \mathcal{P})\,, \tag{69}$$

where $\mathcal{J}(\phi)$ is the only surviving unconstrained component of the conserved current $S(g)$ of the non-gauged co-orbit model. The algebra of $K_A(\phi)$, (68), is the same $G_{\mathrm{stat}}$ Kac-Moody algebra as (41) due to the BRST-invariance of $K_A^{\mathrm{BRST}}(g; \Lambda_H, \mathcal{M}_K, c_H, b_K)$ (cf. (65)). The algebra of the currents $\mathcal{J}(\phi)$ is the Dirac bracket algebra for $c\lambda\,(S(g) - S^{(0)})|_{\mathcal{P}}$ on the physical subspace $\mathcal{O}^*$, (67), and it is given in the standard way[19] in terms of the Poisson bracket relations (25) and (58) (for $B_0 = 0$).

In the particular case where $G$ is the SL($n$; $\mathbb{R}$) Kac-Moody group we have $S(g) = \partial_- g g^{-1}$ and $y_+(g^{-1}) = -g^{-1}\partial_+ g$ with $g(x^+, x^-) \in$ SL($n$; $\mathbb{R}$). Then, as it is well-known $G_{\mathrm{stat}}$ = SL($n$; $\mathbb{R}$), i.e., $\eta_0 = T^A \eta_A(x^+) \in \mathrm{sl}(n; \mathbb{R})$. Let $H$ be the Borel subgroup of the SL($n$; $\mathbb{R}$) Kac-Moody group. Then the Dirac brackets of the currents $\mathcal{J}(\phi)$ exactly coincide[20,21] with the $W_n$ algebra.[22] According to the general analysis above, the resulting $W_n$-geometric action (59) restricted on the physical phase space $\mathcal{O}^*$ (67) automatically inherits the right-handed (with respect to $x^+$) SL($n$; $\mathbb{R}$) Kac-Moody current algebra from the original non-gauged SL($n$; $\mathbb{R}$) WZNW model.

More detailed exposition discussing the moment map interpretation of the basic ingredients of the geometric actions, generalization to semi-direct products (coupling to "matter" fields) and quantum properties of models obtained in the symplectic analysis will be presented elsewhere.

## Acknowledgments

We would like to thank P. B. Wiegmann for useful conversation. We also thank L. A. Ferreira, F. Gomez and A. H. Zimerman for correspondence on the subject. We gratefully acknowledge the cordial hospitality of the Einstein Center for Theoretical Physics and Y. Frishman at the Weizmann Institute of Science, Rehovot and S. Solomon at the Racah Institute of the Hebrew University. H. A. would like to acknowledge support of U. S. -Israel Binational Science Foundation.

## References

1. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333; P. Goddard and D. Olive, *Int. J. Mod. Phys.* **A1** (1986) 303; E. Witten, *Commun. Math. Phys.* **114** (1988) 1.
2. A. A. Kirillov, *Elements of the Theory of Representations* (Springer Verlag, 1976); B. Kostant, *Lect. Notes in Math.* **170** (1970) 87; J. M. Souriau, *Structure des Systems Dynamiques* (Dunod, 1970); R. Abraham and J. Marsden, *Foundations of Mechanics* (Benjamin, 1978); V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge Univ. Press, 1984).
3. A. Alekseev, L. D. Faddeev, and S. L. Shatashvili, *J. Geom. Phys.*, in press.
4. P. B. Wiegmann, *Nucl. Phys.* **B323** (1989) 311.
5. A. Alekseev and S. L. Shatashvili, *Nucl. Phys.* **B323** (1989) 719; *Commun. Math. Phys.* **128** (1990) 197.
6. A. M. Polyakov, *Int. J. Mod. Phys.* **A5** (1990) 833.
7. M. Bershadsky and H. Ooguri. *Commun. Math. Phys.* **126** (1989) 49.
8. V. Drinfeld and V. Sokolov, *J. Sov. Math.* **30** (1984) 1975.
9. A. A. Kirillov., *Lect. Notes in Math.* **970** (1982) 101; *Funct. Anal. and Appl.* **15** (1981) 135; in *Infinite Dimensional Lie Algebras and Quantum Field Theory*, eds. H. D. Doebner *et al.* (World Scientific, 1988) 73.
10. H. Aratyn, E. Nissimov, S. Pacheva, and A. H. Zimerman, *Phys. Lett.* **240B** (1990) 127.
11. A. M. Polyakov and P. B. Wiegmann, *Phys. Lett.* **131B** (1983) 121.
12. H. Aratyn, E. Nissimov, and S. Pacheva, Weizmann preprint WIS-90/24/MAY-PH, to appear in *Phys. Lett.* **B**.
13. K. Schoutens, *Nucl. Phys.* **B295** [FS21] (1988) 634.
14. G. W. Delius, P. van Nieuwenhuizen, and V. G. J. Rodgers, Stony Brook preprint, ITP-SB-89-71, to appear in *Int. J. Mod. Phys.* **A**.
15. S. Aoyama, *Phys. Lett.* **228B** (1989) 355; H. Aratyn, E. Nissimov, S. Pacheva, and S. Solomon, *Phys. Lett.* **234B** (1990) 307.
16. S. Aoyama and J. Julve, *Phys. Lett.* **B243** (1990) 57; R. Rashkov, *Mod. Phys. Lett.* **A5** (1990) 991.
17. H. Aratyn, E. Nissimov, S. Pacheva, and A. H. Zimerman, *Phys. Lett.* **242B** (1990) 377.
18. A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (Polish Sci. Publ., 1980).
19. P. M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva U., New York (1964).
20. J.-D. Smit, *Commun. Math. Phys.* **129** (1990) 1.
21. J. Balog, L. Feher, P. Forgacs, O'Raifeartaigh, and A. Wiff, preprints DIAS-STP-89-31 and MPI-PAE/PTh 9/90.
22. A. B. Zamolodchikov, *Theor. Math. Phys.* **65** (1986) 1205; Fateev and A. B. Zamolodchikov, *Nucl. Phys.* **B280** (1987) 644.